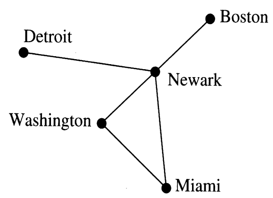
$(A_i \cap A_{i'}) \cap (B_j \cap B_{j'}) = \emptyset$. **25.** The subset relation is a partial ordering on any collection of sets, because it is reflexive, antisymmetric, and transitive. Here the collection of sets is $\mathbf{R}(S)$. **27.** Find recipe $\prec$ Buy seafood $\prec$ Buy groceries $\prec$ Wash shellfish $\prec$ Cut ginger and garlic $\prec$ Clean fish $\prec$ Steam rice $\prec$ Cut fish $\prec$ Wash vegetables $\prec$ Chop water chestnuts $\prec$ Make garnishes $\prec$ Cook in wok $\prec$ Arrange on platter $\prec$ Serve **29. a)** The only antichain with more than one element is $\{c, d\}$. **b)** The only antichains with more than one element are $\{b, c\}$, $\{c, e\}$, and $\{d, e\}$. **c)** The only antichains with more than one element are $\{a, b\}$, $\{a, c\}$, $\{b, c\}$, $\{a, b, c\}$, $\{d, e\}$, $\{d, f\}$, $\{e, f\}$, and $\{d, e, f\}$. **31.** Let $(S, \preccurlyeq)$ be a finite poset, and let A be a maximal chain. Because $(A, \preccurlyeq)$ is also a poset it must have a minimal element m. Suppose that m is not minimal in S. Then there would be an element a of S with $a \prec m$. However, this would make the set $A \cup \{a\}$ a larger chain than A. To show this, we must show that a is comparable with every element of A. Because m is comparable with every element of A and m is minimal, it follows that $m \prec x$ when x is in A and $x \neq m$. Because $a \prec m$ and $m \prec x$, the transitive law shows that $a \prec x$ for every element of A. **33.** Let aRb denote that a is a descendant of b. By Exercise 32, if no set of $n + 1$ people none of whom is a descendant of any other (an antichain) exists, then $k \leq n$, so the set can be partitioned into $k \leq n$ chains. By the pigeonhole principle, at least one of these chains contains at least $m + 1$ people. **35.** We prove by contradiction that if S has no infinite decreasing sequence and $\forall x\,(\{\forall y[y \prec x \to P(y)]\} \to P(x))$, then $P(x)$ is true for all $x \in S$. If it does not hold that $P(x)$ is true for all $x \in S$, let x_1 be an element of S such that $P(x_1)$ is not true. Then by the conditional statement already given, it must be the case that $\forall y[y \prec x_1 \to P(y)]$ is not true. This means that there is some x_2 with $x_2 \prec x_1$ such that $P(x_2)$ is not true. Again invoking the conditional statement, we get an $x_3 \prec x_2$ such that $P(x_3)$ is not true, and so on forever. This contradicts the well-foundedness of our poset. Therefore, $P(x)$ is true for all $x \in S$. **37.** Suppose that R is a quasi-ordering. Because R is reflexive, if $a \in A$, then $(a, a) \in R$. This implies that $(a, a) \in R^{-1}$. Hence, $a \in R \cap R^{-1}$. It follows that $R \cap R^{-1}$ is reflexive. $R \cap R^{-1}$ is symmetric for any relation R because, for any relation R, if $(a, b) \in R$ then $(b, a) \in R^{-1}$ and vice versa. To show that $R \cap R^{-1}$ is transitive, suppose that $(a, b) \in R \cap R^{-1}$ and $(b, c) \in R \cap R^{-1}$. Because $(a, b) \in R$ and $(b, c) \in R$, $(a, c) \in R$, because R is transitive. Similarly, because $(a, b) \in R^{-1}$ and $(b, c) \in R^{-1}$, $(b, a) \in R$ and $(c, b) \in R$, so $(c, a) \in R$ and $(a, c) \in R^{-1}$. Hence, $(a, c) \in R \cap R^{-1}$. It follows that $R \cap R^{-1}$ is an equivalence relation. **39. a)** Because $\text{glb}(x, y) = \text{glb}(y, x)$ and $\text{lub}(x, y) = \text{lub}(y, x)$, it follows that $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$. **b)** Using the definition, $(x \wedge y) \wedge z$ is a lower bound of x, y, and z that is greater than every other lower bound. Because x, y, and z play interchangeable roles, $x \wedge (y \wedge z)$ is the same element. Similarly, $(x \vee y) \vee z$ is an upper bound of x, y, and z that is less than every other upper bound. Because x, y, and z play interchangeable roles, $x \vee (y \vee z)$ is the same element. **c)** To show that $x \wedge (x \vee y) = x$ it is sufficient to show that x is the greatest lower bound of x, and $x \vee y$. Note that x is a lower bound of x, and because $x \vee y$ is by definition greater than x, x is a lower bound for it as well. Therefore, x is a lower bound. But any lower bound of x has to be less than x, so x is the greatest lower bound. The second statement is the dual of the first; we omit its proof. **d)** x is a lower, and an upper, bound for itself and itself, and the greatest, and least, such bound. **41. a)** Because 1 is the only element greater than or equal to 1, it is the only upper bound for 1 and therefore the only possible value of the least upper bound of x and 1. **b)** Because $x \preccurlyeq 1$, x is a lower bound for both x and 1 and no other lower bound can be greater than x, so $x \wedge 1 = x$. **c)** Because $0 \preccurlyeq x$, x is an upper bound for both x and 0 and no other bound can be less than x, so $x \vee 0 = x$. **d)** Because 0 is the only element less than or equal to 0, it is the only lower bound for 0 and therefore the only possible value of the greatest lower bound of x and 0. **43.** $L = (S, \subseteq)$ where $S = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}$ **45.** Yes **47.** The complement of a subset $X \subseteq S$ is its complement $S - X$. To prove this, note that $X \vee (S - X) = 1$ and $X \wedge (S - X) = 0$ because $X \cup (S - X) = S$ and $X \cap (S - X) = \emptyset$. **49.** Think of the rectangular grid as representing elements in a matrix. Thus we number from top to bottom and within that from left to right. The partial order is that $(a, b) \preceq (c, d)$ iff $a \leq c$ and $b \leq d$. Note that $(1, 1)$ is the least element under this relation. The rules for Chomp as explained in Chapter 1 coincide with the rules stated in the preamble here. But now we can identify the point (a, b) with the natural number $p^{a-1}q^{b-1}$ for all a and b with $1 \leq a \leq m$ and $1 \leq b \leq n$. This identifies the points in the rectangular grid with the set S in this exercise, and the partial order $\preceq$ just described is the same as the divides relation, because $p^{a-1}q^{b-1} \mid p^{c-1}q^{d-1}$ if and only if the exponent on p on the left does not exceed the exponent of p on the right, and similarly for q.

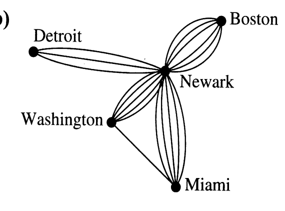
CHAPTER 9

Section 9.1

1. a)



b)



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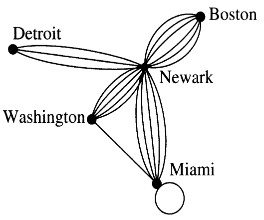
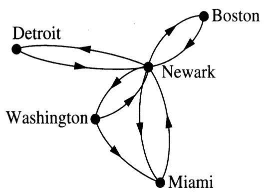
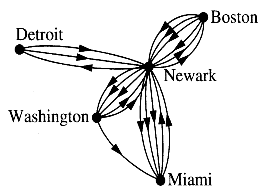
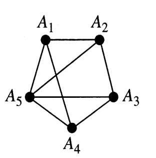


d)


e)


3. Simple graph **5.** Pseudograph **7.** Directed graph **9.** Directed multigraph **11.** If uRv, then there is an edge associated with $\{u, v\}$. But $\{u, v\} = \{v, u\}$, so this edge is associated with $\{v, u\}$ and therefore vRu. Thus, by definition, R is a symmetric relation. A simple graph does not allow loops; therefore, uRu never holds, and so by definition R is irreflexive.

13. a)


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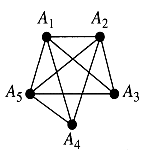
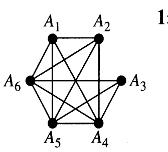


c)


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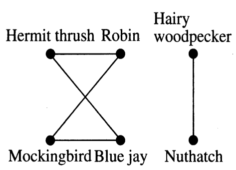

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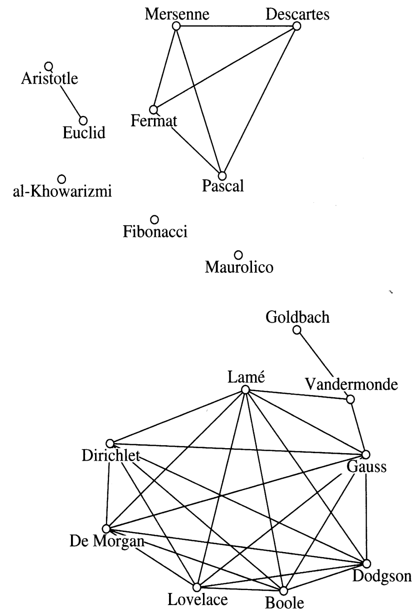

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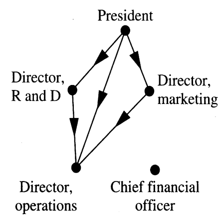

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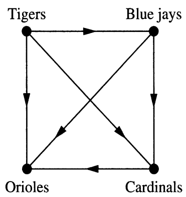
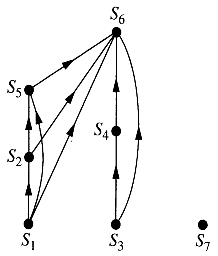


23. We find the telephone numbers in the call graph for February that are not present in the call graph for January and vice versa. For each number we find, we make a list of the numbers they called or were called by using the edges in the call graph. We examine these lists to find new telephone numbers in February that had similar calling patterns to defunct telephone numbers in January. **25.** We use the graph model that has e-mail addresses as vertices and for each message sent, an edge from the e-mail address of the sender to the e-mail address of the recipient. For each e-mail address, we can make a list of other addresses they sent messages to and a list of other address from which they received messages. If two e-mail addresses had almost the same pattern, we conclude that these addresses might have belonged to the same person who had recently changed his or her e-mail address. **27.** Let V be the set of people at the party. Let E be the set of ordered pairs

(u, v) in $V \times V$ such that u knows v's name. The edges are directed, but multiple edges are not allowed. Literally, there is a loop at each vertex, but for simplicity, the model could omit the loops. **29.** Let the set of vertices be a set of people, and two vertices are joined by an edge if the two people were ever married. Ignoring complications, this graph has the property that there are two types of vertices (men and women), and every edge joins vertices of opposite types.

31.

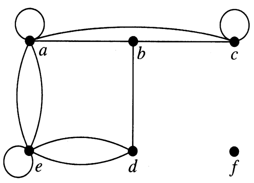


33. Represent people in the group by vertices. Put a directed edge into the graph for every pair of vertices. Label the edge from the vertex representing A to the vertex representing B with a $+$ (plus) if A likes B, a $-$ (minus) if A dislikes B, and a 0 if A is neutral about B.

Section 9.2

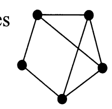
1. $v = 6$; $e = 6$; $\deg(a) = 2$, $\deg(b) = 4$, $\deg(c) = 1$, $\deg(d) = 0$, $\deg(e) = 2$, $\deg(f) = 3$; c is pendant; d is isolated. **3.** $v = 9$; $e = 12$; $\deg(a) = 3$, $\deg(b) = 2$, $\deg(c) = 4$, $\deg(d) = 0$, $\deg(e) = 6$, $\deg(f) = 0$; $\deg(g) = 4$; $\deg(h) = 2$; $\deg(i) = 3$; d and f are isolated. **5.** No, because the sum of the degrees of the vertices cannot be odd **7.** $v = 4$; $e = 7$; $\deg^-(a) = 3$, $\deg^-(b) = 1$, $\deg^-(c) = 2$, $\deg^-(d) = 1$, $\deg^+(a) = 1$, $\deg^+(b) = 2$, $\deg^+(c) = 1$, $\deg^+(d) = 3$ **9.** 5 vertices, 13 edges; $\deg^-(a) = 6$, $\deg^+(a) = 1$, $\deg^-(b) = 1$, $\deg^+(b) = 5$, $\deg^-(c) = 2$, $\deg^+(c) = 5$, $\deg^-(d) = 4$, $\deg^+(d) = 2$, $\deg^-(e) = 0$, $\deg^+(e) = 0$

11.



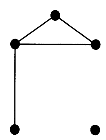
13. The number of collaborators v has; someone who has never collaborated; someone who has just one collaborator **15.** In the directed graph $\deg^-(v) =$ number of calls v received, $\deg^+(v) =$ number of calls v made; in the undirected graph, $\deg(v)$ is the number of calls either made or received by v. **17.** $(\deg^+(v), \deg^-(v))$ is the win–loss record of v. **19.** Construct the simple graph model in which V is the set of people in the group and there is an edge associated with $\{u, v\}$ if u and v know each other. Then the degree of vertex v is the number of people v knows. By the result of Exercise 18, there are two vertices with the same degree. Therefore there are two people who know the same number of other people in the group. **21.** Bipartite **23.** Not bipartite **25.** Not bipartite **27. a)** Let $V = \{$Zamora, Agraharam, Smith, Chou, Macintyre, planning, publicity, sales, marketing, development, industry relations$\}$ and $E = \{\{$Zamora, planning$\}$, $\{$Zamora, sales$\}$, $\{$Zamora, marketing$\}$, $\{$Zamora,industry relations$\}$, $\{$Agraharam, planning$\}$, $\{$Agraharam, development$\}$, $\{$Smith, publicity$\}$, $\{$Smith, sales$\}$, $\{$Smith, industry relations$\}$, $\{$Chou, planning$\}$, $\{$Chou, sales$\}$, $\{$Chou, industry relations$\}$, $\{$Macintyre, planning$\}$, $\{$Macintyre, publicity$\}$, $\{$Macintyre, sales$\}$, $\{$Macintyre, industry relations$\}\}$. **b)** Many answers are possible, such as Zamora–planning, Agraharam–development, Smith–publicity, Chou–sales, and Macintyre–industry relations. **29. a)** n vertices, $n(n-1)/2$ edges **b)** n vertices, n edges **c)** $n + 1$ vertices, $2n$ edges **d)** $m + n$ vertices, mn edges **e)** 2^n vertices, $n2^{n-1}$ edges **31. a)** 3, 3, 3, 3 **b)** 2, 2, 2, 2 **c)** 4,3,3,3,3 **d)** 3,3,2,2,2 **e)** 3, 3, 3, 3, 3, 3, 3, 3 **33.** Each of the n vertices is adjacent to each of the other $n - 1$ vertices, so the degree sequence is $n-1, n-1, \ldots, n-1$ (n terms). **35.** 7

37. a) Yes



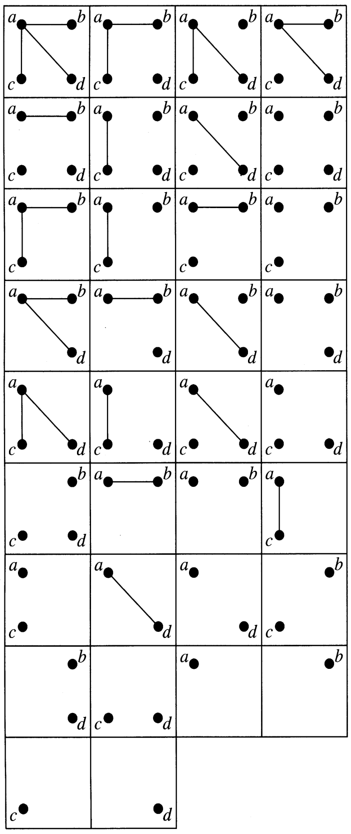
b) No, sum of degrees is odd. **c)** No **d)** No, sum of degrees is odd.

e) Yes



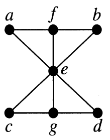
f) No, sum of degrees is odd. **39.** First, suppose that $d_1, d_2, \ldots, d_n$ is graphic. We must show that the sequence whose terms are $d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, d_{d_1+3}, \ldots, d_n$ is graphic once it is put into nonincreasing order. In Exercise 38 it is proved that if the original sequence is graphic, then in fact there is a graph having this degree sequence in which the vertex of degree d_1 is adjacent to the vertices of degrees $d_2, d_3, \ldots, d_{d_1+1}$. Remove from this graph the vertex of highest degree (d_1). The resulting graph has the desired degree sequence. Conversely, suppose that $d_1, d_2, \ldots, d_n$ is a nonincreasing sequence such that the sequence $d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, d_{d_1+3}, \ldots, d_n$ is graphic once it is put into nonincreasing order. Take a graph with this latter degree sequence, where vertex v_i has degree $d_i - 1$ for $2 \le i \le d_1 + 1$ and vertex v_i has degree d_i for $d_1 + 2 \le i \le n$. Adjoin one new vertex (call it v_1), and put in an edge from v_1 to each of the vertices $v_2, v_3, \ldots, v_{d_1+1}$. The resulting graph has degree sequence $d_1, d_2, \ldots, d_n$. **41.** Let $d_1, d_2, \ldots, d_n$ be a nonincreasing sequence of nonnegative integers with an even sum. Construct a graph as follows: Take vertices $v_1, v_2, \ldots, v_n$ and put $\lfloor d_i/2 \rfloor$ loops at vertex v_i, for $i = 1, 2, \ldots, n$. For each i, vertex v_i now has degree either d_i or $d_i - 1$. Because the original sum was even, the number of vertices for which $\deg(v_i) = d_i - 1$ is even. Pair them up arbitrarily, and put in an edge joining the vertices in each pair. **43.** 17

45.



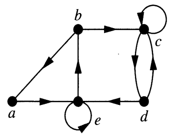
47. a) For all $n \geq 1$ **b)** For all $n \geq 3$ **c)** For $n = 3$ **d)** For all $n \geq 0$ **49.** 5

51.

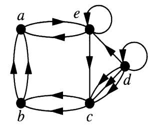


53. a) The graph with n vertices and no edges **b)** The disjoint union of K_m and K_n **c)** The graph with vertices $\{v_1, \ldots, v_n\}$ with an edge between v_i and v_j unless $i \equiv j \pm 1 \pmod{n}$ **d)** The graph whose vertices are represented by bit strings of length n with an edge between two vertices if the associated bit strings differ in more than one bit **55.** $v(v-1)/2 - e$ **57.** $n-1-d_n, n-1-d_{n-1}, \ldots, n-1-d_2, n-1-d_1$ **59.** The union of G and $\overline{G}$ contains an edge between each pair of the n vertices. Hence, this union is K_n.

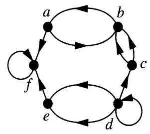
61. Exercise 7:



Exercise 8:

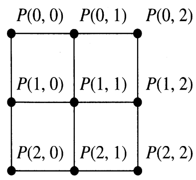


Exercise 9:



63. A directed graph $G = (V, E)$ is its own converse if and only if it satisfies the condition $(u, v) \in E$ if and only if $(v, u) \in E$. But this is precisely the condition that the associated relation must satisfy to be symmetric.

65.



67. We can connect $P(i, j)$ and $P(k, l)$ by using $|i - k|$ hops to connect $P(i, j)$ and $P(k, j)$ and $|j - l|$ hops to connect $P(k, j)$ and $P(k, l)$. Hence, the total number of hops required to connect $P(i, j)$ and $P(k, l)$ does not exceed $|i - k| + |j - l|$. This is less than or equal to $m + m = 2m$, which is $O(m)$.

Section 9.3

1.

Vertex	Adjacent Vertices
a	b, c, d
b	a, d
c	a, d
d	a, b, c

3.

Vertex	Terminal Vertices
a	a, b, c, d
b	d
c	a, b
d	b, c, d

5. $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

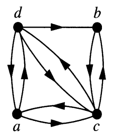
where vertices are listed in alphabetical order

7. $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$ **9. a)** $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

b) $\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$ **c)** $\begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}$

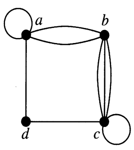
d) $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$ **e)** $\begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$

f) $\begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$

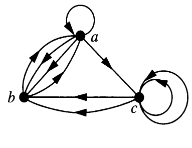
11. 

13. $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix}$

15. $\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}$

17. 

19. $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ **21.** $\begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 0 & 0 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix}$

23. 

25. Yes

27. Exercise 13: $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$

Exercise 14: $\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$

Exercise 15: $\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$

29. $\deg(v)$ − number of loops at v; $\deg^-(v)$ **31.** 2 if e is not a loop, 1 if e is a loop

33. a) $\begin{bmatrix} 1 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 1 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 1 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 0 & \cdots & 0 & 1 \\ 1 & 1 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 1 & 1 \end{bmatrix}$

c) $\begin{bmatrix} 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ & & & & 1 & 0 & \cdots & 0 \\ & & \mathbf{B} & & 0 & 1 & \cdots & 0 \\ & & & & \vdots & \vdots & & \vdots \\ & & & & 0 & 0 & \cdots & 1 \end{bmatrix}$

where **B** is the answer to (b)

d) $\begin{bmatrix} 1 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 1 \\ 1 & 0 & & 0 & 1 & \cdots & 0 \\ 0 & 1 & & 0 & 0 & & \\ \vdots & \vdots & & \vdots & \vdots & & \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 1 \end{bmatrix}$

35. Isomorphic **37.** Isomorphic **39.** Isomorphic **41.** Not isomorphic **43.** Isomorphic **45.** G is isomorphic to itself by the identity function, so isomorphism is reflexive. Suppose that G is isomorphic to H. Then there exists a one-to-one correspondence f from G to H that preserves adjacency and nonadjacency. It follows that f^{-1} is a one-to-one correspondence from H to G that preserves adjacency and nonadjacency. Hence, isomorphism is symmetric. If G is isomorphic to H and H is isomorphic to K, then there are one-to-one correspondences f and g from G to H and from H to K that preserve adjacency and nonadjacency. It follows that $g \circ f$ is a one-to-one correspondence from G to K that preserves adjacency and nonadjacency. Hence, isomorphism is transitive. **47.** All zeros **49.** Label the vertices in order so that all of the vertices in the first set of the partition of the vertex set come first. Because no edges join vertices in the same set of the partition, the matrix has the desired form. **51.** C_5 **53.** $n = 5$ only **55.** 4 **57. a)** Yes **b)** No **c)** No **59.** $G = (V_1, E_1)$ is isomorphic to $H =$

(V_2, E_2) if and only if there exist functions f from V_1 to V_2 and g from E_1 to E_2 such that each is a one-to-one correspondence and for every edge e in E_1 the endpoints of $g(e)$ are $f(v)$ and $f(w)$ where v and w are the endpoints of e. **61.** Yes **63.** Yes **65.** If f is an isomorphism from a directed graph G to a directed graph H, then f is also an isomorphism from G^{conv} to H^{conv}. To see this note that (u, v) is an edge of G^{conv} if and only if (v, u) is an edge of G if and only if $(f(v), f(u))$ is an edge of H if and only if $(f(u), f(v))$ is an edge of H^{conv}. **67.** Many answers are possible; for example, C_6 and $C_3 \cup C_3$. **69.** The product is $[a_{ij}]$ where a_{ij} is the number of edges from v_i to v_j when $i \neq j$ and a_{ii} is the number of edges incident to v_i. **71.** The graphs in Exercise 41 provide a devil's pair.

Section 9.4

1. a) Path of length 4; not a circuit; not simple **b)** Not a path **c)** Not a path **d)** Simple circuit of length 5 **3.** No **5.** No **7.** Maximal sets of people with the property that for any two of them, we can find a string of acquaintances that takes us from one to the other **9.** If a person has Erdős number n, then there is a path of length n from that person to Erdős in the collaboration graph, so by definition, that means that that person is in the same component as Erdős. If a person is in the same component as Erdős, then there is a path from that person to Erdős, and the length of the shortest such path is that person's Erdős number. **11. a)** Weakly connected **b)** Weakly connected **c)** Not strongly or weakly connected **13.** The maximal sets of phone numbers for which it is possible to find directed paths between every two different numbers in the set **15. a)** $\{a, b, f\}$, $\{c, d, e\}$ **b)** $\{a, b, c, d, e, h\}$, $\{f\}$, $\{g\}$ **c)** $\{a, b, d, e, f, g, h, i\}$, $\{c\}$ **17. a)** 2 **b)** 7 **c)** 20 **d)** 61 **19.** Not isomorphic (G has a triangle; H does not) **21.** Isomorphic (the path $u_1, u_2, u_7, u_6, u_5, u_4, u_3, u_8, u_1$ corresponds to the path $v_1, v_2, v_3, v_4, v_5, v_8, v_7, v_6, v_1$) **23. a)** 3 **b)** 0 **c)** 27 **d)** 0 **25. a)** 1 **b)** 0 **c)** 2 **d)** 1 **e)** 5 **f)** 3 **27.** R is reflexive by definition. Assume that $(u, v) \in R$; then there is a path from u to v. Then $(v, u) \in R$ because there is a path from v to u, namely, the path from u to v traversed backward. Assume that $(u, v) \in R$ and $(v, w) \in R$; then there are paths from u to v and from v to w. Putting these two paths together gives a path from u to w. Hence, $(u, w) \in R$. It follows that R is transitive. **29.** c **31.** b, c, e, i **33.** If a vertex is pendant it is clearly not a cut vertex. So an endpoint of a cut edge that is a cut vertex is not pendant. Removal of a cut edge produces a graph with more connected components than in the original graph. If an endpoint of a cut edge is not pendant, the connected component it is in after the removal of the cut edge contains more than just this vertex. Consequently, removal of that vertex and all edges incident to it, including the original cut edge, produces a graph with more connected components than were in the original graph. Hence, an endpoint of a cut edge that is not pendant is a cut vertex. **35.** Assume there exists a connected graph G with at most one vertex that is not a cut vertex. Define the distance between the vertices u and v, denoted by $d(u, v)$, to be the length of the shortest path between u and v in G. Let s and t be vertices in G such that $d(s, t)$ is a maximum. Either s or t (or both) is a cut vertex, so without loss of generality suppose that s is a cut vertex. Let w belong to the connected component that does not contain t of the graph obtained by deleting s and all edges incident to s from G. Because every path from w to t contains s, $d(w, t) > d(s, t)$, which is a contradiction. **37. a)** Denver–Chicago, Boston–New York **b)** Seattle–Portland, Portland–San Francisco, Salt Lake City–Denver, New York–Boston, Boston–Burlington, Boston–Bangor **39.** A set of people who collectively influence everyone (directly or indirectly); {Deborah} **41.** An edge cannot connect two vertices in different connected components. Because there are at most $C(n_i, 2)$ edges in the connected component with n_i vertices, it follows that there are at most $\sum_{i=1}^{k} C(n_i, 2)$ edges in the graph. **43.** Suppose that G is not connected. Then it has a component of k vertices for some k, $1 \leq k \leq n - 1$. The most edges G could have is $C(k, 2) + C(n - k, 2) = [k(k - 1) + (n - k)(n - k - 1)]/2 = k^2 - nk + (n^2 - n)/2$. This quadratic function of f is minimized at $k = n/2$ and maximized at $k = 1$ or $k = n - 1$. Hence, if G is not connected, the number of edges does not exceed the value of this function at 1 and at $n - 1$, namely, $(n - 1)(n - 2)/2$. **45. a)** 1 **b)** 2 **c)** 6 **d)** 21 **47.** 2 **49.** Let the paths P_1 and P_2 be $u = x_0, x_1, \ldots, x_n = v$, respectively. Because P_1 and P_2 do not contain the same set of edges, they must eventually diverge. If this happens only after one of them has ended, the rest of the other path is a simple circuit from v to v. Otherwise, we can suppose that $x_0 = y_0, x_1 = y_1, \ldots, x_i = y_i$, but $x_{i+1} \neq y_{i+1}$. Follow the path y_i, y_{i+1}, y_{i+2}, and so on, until it once again encounters a vertex on P_1. Once we are back on P_1, follow it, forward or backward as necessary, back to x_i. Because $x_i = y_i$, this forms a circuit that must be simple, because no edge among the x_ks can be repeated and no edge among the x_ks can equal one of the Y_ls that we used. **51.** The graph G is connected if and only if every off-diagonal entry of $\mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3 + \cdots + \mathbf{A}^{n-1}$ is positive, where $\mathbf{A}$ is the adjacency matrix of G. **53.** If the graph is bipartite, say with parts A and B, then the vertices in every path must alternately lie in A and B. Therefore a path that starts in A, say, will end in B after an odd number of steps and in A after an even number of steps. Because a circuit ends at the same vertex where it starts, the length must be even. Conversely, suppose that all circuits have even length; we must show that the graph is bipartite. We can assume that the graph is connected, because if it is not, then we can just work on one component at a time. Let v be a vertex of the graph, and let A be the set of all vertices to which there is a path of odd length starting at v, and let B be the set of all vertices to which there is a path of even length starting at v. Because the component is connected, every vertex lies in A or B. No vertex can lie in both A and B, because if one did, then following the odd-length path from v to that vertex and then back along the even-length path from that vertex to v would produce an odd circuit, contrary to the hypothesis. Thus, the set of vertices has been partitioned into two sets. To show that every edge has endpoints in different parts, suppose that xy is an edge, where $x \in A$. Then the odd-length path from v to x followed by xy produces an even-length path from v to y, so $y \in B$. (Similarly, if $x \in B$.) **55.** $(H_1W_1H_2W_2\langle\text{boat}\rangle, \emptyset) \rightarrow (H_2W_2, H_1W_1\langle\text{boat}\rangle) \rightarrow (H_1H_2W_2\langle\text{boat}\rangle, W_1) \rightarrow (W_2, H_1W_1H_2\langle\text{boat}\rangle) \rightarrow (H_2W_2\langle\text{boat}\rangle, H_1W_1) \rightarrow (\emptyset, H_1W_1H_2W_2\langle\text{boat}\rangle)$

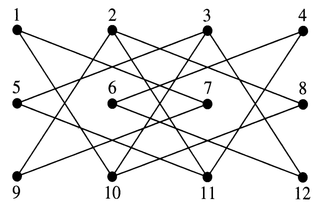
Section 9.5

1. Neither **3.** No Euler circuit; $a, e, c, e, b, e, d, b, a, c, d$ **5.** $a, b, c, d, c, e, d, b, e, a, e, a$ **7.** $a, i, h, g, d, e, f, g, c, e, h, d, c, a, b, i, c, b, h, a$ **9.** No, A still has odd degree. **11.** When the graph in which vertices represent intersections and edges streets has an Euler path **13.** Yes **15.** No **17.** If there is an Euler path, as we follow it each vertex except the starting and ending vertices must have equal in-degree and out-degree, because whenever we come to a vertex along an edge, we leave it along another edge. The starting vertex must have out-degree 1 larger than its in-degree, because we use one edge leading out of this vertex and whenever we visit it again we use one edge leading into it and one leaving it. Similarly, the ending vertex must have in-degree 1 greater than its out-degree. Because the Euler path with directions erased produces a path between any two vertices, in the underlying undirected graph, the graph is weakly connected. Conversely, suppose the graph meets the degree conditions stated. If we add one more edge from the vertex of deficient out-degree to the vertex of deficient in-degree, then the graph has every vertex with equal in-degree and out-degree. Because the graph is still weakly connected, by Exercise 16 this new graph has an Euler circuit. Now delete the added edge to obtain the Euler path. **19.** Neither **21.** No Euler circuit; $a, d, e, d, b, a, e, c, e, b, c, b, e$ **23.** Neither **25.** Follow the same procedure as Algorithm 1, taking care to follow the directions of edges. **27. a)** $n = 2$ **b)** None **c)** None **d)** $n = 1$ **29.** Exercise 1:1 time; Exercises 2–7: 0 times **31.** a, b, c, d, e, a is a Hamilton circuit. **33.** No Hamilton circuit exists, because once a purported circuit has reached e it would have nowhere to go. **35.** No Hamilton circuit exists, because every edge in the graph is incident to a vertex of degree 2 and therefore must be in the circuit. **37.** a, b, c, f, d, e is a Hamilton path. **39.** f, e, d, a, b, c is a Hamilton path. **41.** No Hamilton path exists. There are eight vertices of degree 2, and only two of them can be end vertices of a path. For each of the other six, their two incident edges must be in the path. It is not hard to see that if there is to be a Hamilton path, exactly one of the inside corner vertices must be an end, and that this is impossible. **43.** $a, b, c, f, i, h, g, d, e$ is a Hamilton path. **45.** $m = n \geq 2$ **47. a)** (i) No, (ii) No, (iii) Yes **b)** (i) No, (ii) No, (iii) Yes **c)** (i) Yes, (ii) Yes, (iii) Yes **d)** (i) Yes, (ii) Yes, (iii) Yes **49.** The result is trivial for $n = 1$: code is 0, 1. Assume we have a Gray code of order n. Let $c_1, \ldots, c_k$, $k = 2^n$ be such a code. Then $0c_1, \ldots, 0c_k, 1c_k, \ldots, 1c_1$ is a Gray code of order $n + 1$.

51.
```
procedure Fleury (G = (V, E): connected multigraph
    with the degrees of all vertices even, V = {v1, ..., vn})
v := v1
circuit := v
H := G
while H has edges
begin
    e := first edge with endpoint v in H (with respect
      to listing of V) such that e is not a cut edge of H, if
      one exists, and simply the first edge in H with
      endpoint v otherwise
    w := other endpoint of e
    circuit := circuit with e, w added
    v := w
    H := H − e
end {circuit is an Euler circuit}
```

53. If G has an Euler circuit, it is also an Euler path. If not, add an edge between the two vertices of odd degree and apply the algorithm to get an Euler circuit. Then delete the new edge. **55.** Suppose $G = (V, E)$ is a bipartite graph with $V = V_1 \cup V_2$, where no edge connects a vertex in V_1 and a vertex in V_2. Suppose that G has a Hamilton circuit. Such a circuit must be of the form $a_1, b_1, a_2, b_2, \ldots, a_k, b_k, a_1$, where $a_i \in V_1$ and $b_i \in V_2$ for $i = 1, 2, \ldots, k$. Because the Hamilton circuit visits each vertex exactly once, except for v_1, where it begins and ends, the number of vertices in the graph equals $2k$, an even number. Hence, a bipartite graph with an odd number of vertices cannot have a Hamilton circuit.

57.



59. We represent the squares of a 3×4 chessboard as follows:

1	2	3	4
5	6	7	8
9	10	11	12

A knight's tour can be made by following the moves 8, 10, 1, 7, 9, 2, 11, 5, 3, 12, 6, 4. **61.** We represent the squares of a 4×4 chessboard as follows:

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

There are only two moves from each of the four corner squares. If we include all the edges 1–10, 1–7, 16–10, and 16–7, a circuit is completed too soon, so at least one of these edges must be missing. Without loss of generality, assume the path starts 1–10, 10–16, 16–7. Now the only moves from square 3 are to squares 5, 10, and 12, and square 10 already has two incident edges. Therefore, 3–5 and 3–12 must be in the Hamilton circuit. Similarly, edges 8–2 and 8–15 must be in the circuit. Now the only moves from square 9 are to squares 2, 7, and 15. If there were edges from square 9 to both squares 2 and 15, a circuit would be completed too soon. Therefore the edge 9–7 must be in the circuit giving square 7 its full complement of edges. But now square 14 is forced to be joined to squares 5 and 12, completing a circuit too soon (5–14–12–3–5). This contradiction shows that there is no knight's tour on the 4×4 board. **63.** Because there are mn squares on an $m \times n$ board, if both m and n are odd, there are an odd number of squares. Because by Exercise 62 the corresponding graph is bipartite, by Exercise 55 it has no Hamilton circuit. Hence, there is no reentrant knight's tour. **65. a)** If G does not have a Hamilton circuit, continue as long as possible adding missing edges one at a time in such a way that we do not obtain a graph with a Hamilton circuit. This cannot go on forever, because once we've formed the complete graph by adding all missing edges, there is a Hamilton circuit. Whenever the process stops, we have obtained a (necessarily noncomplete) graph H with the desired property. **b)** Add one more edge to H. This produces a Hamilton circuit, which uses the added edge. The path consisting of this circuit with the added edge omitted is a Hamilton path in H. **c)** Clearly v_1 and v_n are not adjacent in H, because H has no Hamilton circuit. Therefore they are not adjacent in G. But the hypothesis was that the sum of the degrees of vertices not adjacent in G was at least n. This inequality can be rewritten as $n - \deg(v_n) \le \deg(v_1)$. But $n - \deg(v_n)$ is just the number of vertices not adjacent to v_n. **d)** Because there is no vertex following v_n in the Hamilton path, v_n is not in S. Each one of the $\deg(v_1)$ vertices adjacent to v_1 gives rise to an element of S, so S contains $\deg(v_1)$ vertices. **e)** By part (c) there are at most $\deg(v_1) - 1$ vertices other than v_n not adjacent to v_n, and by part (d) there are $\deg(v_1)$ vertices in S, none of which is v_n. Therefore at least one vertex of S is adjacent to v_n. By definition, if v_k is this vertex, then H contains edges $v_k v_n$ and $v_1 v_{k+1}$, where $1 < k < n - 1$. **f)** Now $v_1, v_2, \ldots, v_{k-1}, v_k, v_n, v_{n-1}, \ldots, v_{k+1}, v_1$ is a Hamilton circuit in H, contradicting the construction of H. Therefore, our assumption that G did not originally have a Hamilton circuit is wrong, and our proof by contradiction is complete.

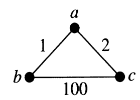
Section 9.6

1. a) Vertices are the stops, edges join adjacent stops, weights are the times required to travel between adjacent stops. **b)** Same as part (a), except weights are distances between adjacent stops. **c)** Same as part (a), except weights are fares between stops. **3.** 16 **5.** Exercise 2: *a, b, e, d, z*; Exercise 3: *a, c, d, e, g, z*; Exercise 4: *a, b, e, h, l, m, p, s, z* **7. a)** *a, c, d* **b)** *a, c, d, f* **c)** *c, d, f* **e)** *b, d, e, g, z* **9. a)** Direct **b)** Via New York **c)** Via Atlanta and Chicago **d)** Via New York **11. a)** Via Chicago **b)** Via Chicago **c)** Via Los Angeles **d)** Via Chicago **13. a)** Via Chicago **b)** Via Chicago **c)** Via Los Angeles **d)** Via Chicago **15.** Do not stop the algorithm when z is added to the set S. **17. a)** Via Woodbridge, via Woodbridge and Camden **b)** Via Woodbridge, via Woodbridge and Camden **19.** For instance, sightseeing tours, street cleaning

21.

	a	***b***	***c***	***d***	***e***	***z***
a	4	3	2	8	10	13
b	3	2	1	5	7	10
c	2	1	2	6	8	11
d	8	5	6	4	2	5
e	10	7	8	2	4	3
z	13	10	11	5	3	6

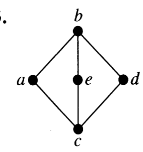
23. $O(n^3)$ **25.** *a-c-b-d-a* (or the same circuit starting at some point but traversing the vertices in the same or reverse order) **27.** San Francisco–Denver–Detroit–New York–Los Angeles–San Francisco (or the same circuit starting at some other point but traversing the vertices in the same or reverse order)
29. Consider this graph:



The circuit *a-b-a-c-a* visits each vertex at least once (and the vertex a twice) and has total weight 6. Every Hamilton circuit has total weight 103.

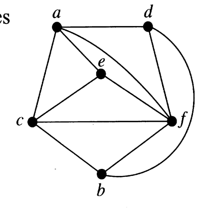
Section 9.7

1. Yes **3.**



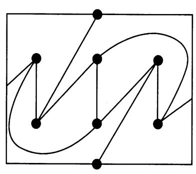
5. No

7. Yes



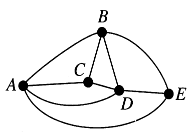
9. No **11.** A triangle is formed by the planar representation of the subgraph of K_5 consisting of the edges connecting v_1, v_2, and v_3. The vertex v_4 must be placed either within the triangle or outside of it. We will consider only the case when v_4 is inside the triangle; the other case is similar. Drawing the three edges from v_1, v_2, and v_3 to v_4 forms four regions. No matter which of these four regions v_5 is in, it is possible to join it to only three, and not all four, of the other vertices. **13.** 8 **15.** Because there are no loops or multiple edges and no simple circuits of length 3, and the degree of the unbounded region is at least 4, each region has degree at least 4. Thus $2e \geq 4r$, or $r \leq e/2$. But $r = e - v + 2$, so we have $e - v + 2 \leq e/2$, which implies that $e \leq 2v - 4$. **17.** As in the argument in the proof of Corollary 1, we have $2e \geq 5r$ and $r = e - v + 2$. Thus $e - v + 2 \leq 2e/5$, which implies that $e \leq (5/3)v - (10/3)$. **19.** Only (a) and (c) **21.** Not homeomorphic to $K_{3,3}$ **23.** Planar **25.** Nonplanar **27. a)** 1 **b)** 3 **c)** 9 **d)** 2 **e)** 4 **f)** 16 **29.** Draw $K_{m,n}$ as described in the hint. The number of crossings is four times the number in the first quadrant. The vertices on the x-axis to the right of the origin are $(1, 0), (2, 0), \ldots, (m/2, 0)$ and the vertices on the y-axis above the origin are $(0, 1), (0, 2), \ldots, (0, n/2)$. We obtain all crossings by choosing any two numbers a and b with $1 \leq a < b \leq m/2$ and two numbers r and s with $1 \leq r < s \leq n/2$; we get exactly one crossing in the graph between the edge connecting $(a, 0)$ and $(0, s)$ and the edge connecting $(b, 0)$ and $(0, r)$. Hence, the number of crossings in the first quadrant is $C\left(\frac{m}{2}, 2\right) \cdot C\left(\frac{n}{2}, 2\right) = \frac{(m/2)(m/2-1)}{2} \cdot \frac{(n/2)(n/2-1)}{2}$. Hence, the total number of crossings is $4 \cdot mn(m-2)(n-2)/64 = mn(m-2)(n-2)/16$. **31. a)** 2 **b)** 2 **c)** 2 **d)** 2 **e)** 2 **f)** 2 **33.** The formula is valid for $n \leq 4$. If $n > 4$, by Exercise 32 the thickness of K_n is at least $C(n, 2)/(3n - 6) = (n + 1 + \frac{2}{n-2})/6$ rounded up. Because this quantity is never an integer, it equals the next larger integer rounded down, which equals $\lfloor (n + 7)/6 \rfloor$. **35.** This follows from Exercise 34 because $K_{m,n}$ has mn edges and $m + n$ vertices and has no triangles because it is bipartite.

37.

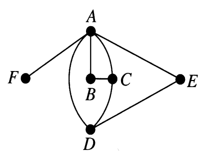


Section 9.8

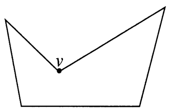
1. Four colors



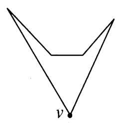
3. Three colors



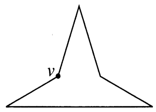
5. 3 **7.** 3 **9.** 2 **11.** 3 **13.** Graphs with no edges **15.** 3 if n is even, 4 if n is odd **17.** Period 1: Math 115, Math 185; period 2: Math 116, CS 473; period 3: Math 195, CS 101; period 4: CS 102; period 5: CS 273 **19.** 5 **21.** Exercise 5: 3 Exercise 6: 6 Exercise 7: 3 Exercise 8: 4 Exercise 9: 3 Exercise 10: 6 Exercise 11: 4 **23.** 5 **25.** Color 1: e, f, d; color 2: c, a, i, g; color 3: h, b, j **27.** Color C_6 **29.** Four colors are needed to color W_n when n is an odd integer greater than 1, because three colors are needed for the rim (see Example 4), and the center vertex, being adjacent to all the rim vertices, will require a fourth color. To see that the graph obtained from W_n by deleting one edge can be colored with three colors, consider two cases. If we remove a rim edge, then we can color the rim with two colors, by starting at an endpoint of the removed edge and using the colors alternately around the portion of the rim that remains. The third color is then assigned to the center vertex. If we remove a spoke edge, then we can color the rim by assigning color #1 to the rim endpoint of the removed edge and colors #2 and #3 alternately to the remaining vertices on the rim, and then assign color #1 to the center. **31.** Suppose that G is chromatically k-critical but has a vertex v of degree $k - 2$ or less. Remove from G one of the edges incident to v. By definition of "k-critical," the resulting graph can be colored with $k - 1$ colors. Now restore the missing edge and use this coloring for all vertices except v. Because we had a proper coloring of the smaller graph, no two adjacent vertices have the same color. Furthermore, v has at most $k - 2$ neighbors, so we can color v with an unused color to obtain a proper $(k - 1)$-coloring of G. This contradicts the fact that G has chromatic number k. Therefore, our assumption was wrong, and every vertex of G must have degree at least $k - 1$. **33. a)** 6 **b)** 7 **c)** 9 **d)** 11 **35.** Represent frequencies by colors and zones by vertices. Join two vertices with an edge if the zones these vertices represent interfere with one another. Then a k-tuple coloring is precisely an assignment of frequencies that avoids interference. **37.** We use induction on the number of vertices of the graph. Every graph with five or fewer vertices can be colored with five or fewer colors, because each vertex can get a different color. That takes care of the basis case(s). So we assume that all graphs with k vertices can be 5-colored and consider a graph G with $k + 1$ vertices. By Corollary 2 in Section 9.7, G has a vertex v with degree at most 5. Remove v to form the graph G'. Because G' has only k vertices, we 5-color it by the inductive hypothesis. If the neighbors of v do not use all five colors, then we can 5-color G by assigning to v a color not used by any of its neighbors. The difficulty arises if v has five neighbors, and each has a different color in the 5-coloring of G'. Suppose that the neighbors of v, when considered in clockwise order around v, are a, b, c, m, and p. (This order is determined by the clockwise order of the curves representing the edges incident to v.) Suppose that the colors of the neighbors are azure, blue, chartreuse, magenta, and purple, respectively. Consider the azure-chartreuse subgraph (i.e., the vertices in G colored azure or chartreuse and all the edges between them). If a and c are not in the same component of this graph, then in the component containing a we can interchange these two colors (make the azure vertices chartreuse and vice versa), and G' will still be properly colored. That makes a chartreuse, so we can now color v azure, and G has been properly colored. If a and c are in the same component, then there is a path of vertices alternately colored azure and chartreuse joining a and c. This path together with edges av and vc divides the plane into two regions, with b in one of them and m in the other. If we now interchange blue and magenta on all the vertices in the same region as b, we will still have a proper coloring of G', but now blue is available for v. In this case, too, we have found a proper coloring of G. This completes the inductive step, and the theorem is proved. **39.** We follow the hint. Because the measures of the interior angles of a pentagon total 540°, there cannot be as many as three interior angles of measure more than 180° (reflex angles). If there are no reflex angles, then the pentagon is convex, and a guard placed at any vertex can see all points. If there is one reflex angle, then the pentagon must look essentially like figure (a) below, and a guard at vertex v can see all points. If there are two reflex angles, then they can be adjacent or nonadjacent (figures (b) and (c)); in either case, a guard at vertex v can see all points. [In figure (c), choose the reflex vertex closer to the bottom side.] Thus for all pentagons, one guard suffices, so $g(5) = 1$.



(a)



(b)



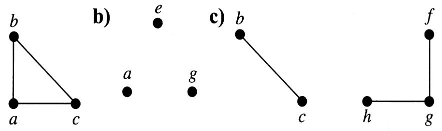
(c)

41. The figure suggested in the hint (generalized to have k prongs for any $k \geq 1$) has $3k$ vertices. The sets of locations from which the tips of different prongs are visible are disjoint. Therefore, a separate guard is needed for each of the k prongs, so at least k guards are needed. This shows that $g(3k) \geq k = \lfloor 3k/3 \rfloor$. If $n = 3k + i$, where $0 \leq i \leq 2$, then $g(n) \geq g(3k) \geq k = \lfloor n/3 \rfloor$.

Supplementary Exercises

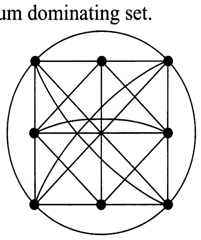
1. 2500 **3.** Yes **5.** Yes **7.** $\sum_{i=1}^{m} n_i$ vertices, $\sum_{i<j} n_i n_j$ edges

9. a) **b)** **c)**

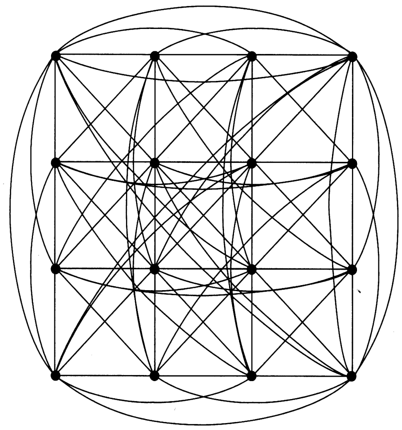


11. Complete subgraphs containing the following sets of vertices: $\{b, c, e, f\}$, $\{a, b, g\}$, $\{a, d, g\}$, $\{d, e, g\}$, $\{b, e, g\}$ **13.** Complete subgraphs containing the following sets of vertices: $\{b, c, d, j, k\}$, $\{a, b, j, k\}$, $\{e, f, g, i\}$, $\{a, b, i\}$, $\{a, i, j\}$, $\{b, d, e\}$, $\{b, e, i\}$, $\{b, i, j\}$, $\{g, h, i\}$, $\{h, i, j\}$ **15.** $\{c, d\}$ is a minimum dominating set.

17. a)



b)



19. a) 1 **b)** 2 **c)** 3 **21. a)** A path from u to v in a graph G induces a path from $f(u)$ to $f(v)$ in an isomorphic graph H. **b)** Suppose f is an isomorphism from G to H. If $v_0, v_1, \ldots, v_n, v_0$ is a Hamilton circuit in G, then $f(v_0), f(v_1), \ldots, f(v_n), f(v_0)$ must be a Hamilton circuit in H because it is still a circuit and $f(v_i) \neq f(v_j)$ for $0 \leq i < j \leq n$. **c)** Suppose f is an isomorphism from G to H. If $v_0, v_1, \ldots, v_n, v_0$ is an Euler circuit in G, then $f(v_0), f(v_1), \ldots, f(v_n), f(v_0)$ must be an Euler circuit in H because it is a circuit that contains each edge exactly once. **d)** Two isomorphic graphs must have the same crossing number because they can be drawn exactly the same way in the plane. **e)** Suppose f is an isomorphism from G to H. Then v is isolated in G if and only if $f(v)$ is isolated in H. Hence, the graphs must have the same number of isolated vertices. **f)** Suppose f is an isomorphism from G to H. If G is bipartite, then the vertex set of G can be partitioned into V_1 and V_2 with no edge connecting a vertex of V_1 with one in V_2. Then the vertex set of H can be partitioned into $f(V_1)$ and $f(V_2)$ with no edge connecting a vertex in $f(V_1)$ and one in $f(V_2)$. **23.** 3 **25. a)** Yes **b)** No **27.** No **29.** Yes **31.** If e is a cut edge with endpoints u and v, then if we direct e from u to v, there will be no path in the directed graph from v to u, or else e would not have been a cut edge. Similar reasoning works if we direct e from v to u. **33.** $n - 1$ **35.** Let the vertices represent the chickens. We include the edge (u, v) in the graph if and only if chicken u dominates chicken v. **37. a)** 4 **b)** 2 **c)** 3 **d)** 4 **e)** 4 **f)** 2 **39. a)** Suppose that $G = (V, E)$. Let $a, b \in V$. We must show that the distance between a and b in $\overline{G}$ is at most 2. If $\{a, b\} \notin E$ this distance is 1, so assume $\{a, b\} \in E$. Because the diameter of G is greater than 3, there are vertices u and v such that the distance in G between u and v is greater than 3. Either u or v, or both, is not in the set $\{a, b\}$. Assume that u is different from both a and b. Either $\{a, u\}$ or $\{b, u\}$ belongs to E; otherwise a, u, b would be a path in $\overline{G}$ of length 2. So, without loss of generality, assume $\{a, u\} \in E$. Thus v cannot be a or b, and by the same reasoning either $\{a, v\} \in E$ or $\{b, v\} \in E$. In either case, this gives a path of length less than or equal to 3 from

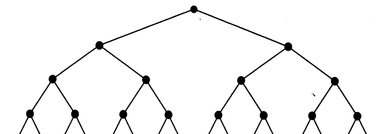
u to v in G, a contradiction. **b)** Suppose $G = (V, E)$. Let $a, b \in V$. We must show that the distance between a and b in $\overline{G}$ does not exceed 3. If $\{a, b\} \notin E$, the result follows, so assume that $\{a, b\} \in E$. Because the diameter of G is greater than or equal to 3, there exist vertices u and v such that the distance in G between u and v is greater than or equal to 3. Either u or v, or both, is not in the set $\{a, b\}$. Assume u is different from both a and b. Either $\{a, u\} \in E$ or $\{b, u\} \in E$; otherwise a, u, b is a path of length 2 in $\overline{G}$. So, without loss of generality, assume $\{a, u\} \in E$. Thus v is different from a and from b. If $\{a, v\} \in E$, then u, a, v is a path of length 2 in G, so $\{a, v\} \notin E$ and thus $\{b, v\} \in E$ (or else there would be a path a, v, b of length 2 in $\overline{G}$). Hence, $\{u, b\} \notin E$; otherwise u, b, v is a path of length 2 in G. Thus, a, v, u, b is a path of length 3 in $\overline{G}$, as desired. **41.** a, b, e, z **43.** a, c, b, d, e, z **45.** If G is planar, then because $e \le 3v - 6$, G has at most 27 edges. (If G is not connected it has even fewer edges.) Similarly, $\overline{G}$ has at most 27 edges. But the union of G and $\overline{G}$ is K_{11}, which has 55 edges, and $55 > 27 + 27$. **47.** Suppose that G is colored with k colors and has independence number i. Because each color class must be an independent set, each color class has no more than i elements. Thus there are at most ki vertices. **49. a)** By Theorem 2 of Section 6.2, the probability of selecting exactly m edges is $C(n, m)p^m(1-p)^{n-m}$. **b)** By Theorem 2 in Section 6.4, the expected value is np. **c)** To generate a labeled graph G, as we apply the process to pairs of vertices, the random number x chosen must be less than or equal to $1/2$ when G has an edge between that pair of vertices and greater than $1/2$ when G has no edge there. Hence, the probability of making the correct choice is $1/2$ for each edge and $1/2^{C(n,2)}$ overall. Hence, all labeled graphs are equally likely. **51.** Suppose P is monotone increasing. If the property of not having P were not retained whenever edges are removed from a simple graph, there would be a simple graph G not having P and another simple graph G' with the same vertices but with some of the edges of G missing that has P. But P is monotone increasing, so because G' has P, so does G obtained by adding edges to G'. This is a contradiction. The proof of the converse is similar.

CHAPTER 10

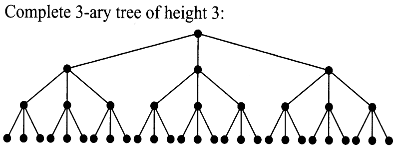
Section 10.1

1. (a), (c), (e) **3. a)** a **b)** $a, b, c, d, f, h, j, q, t$ **c)** $e, g, i, k, l, m, n, o, p, r, s, u$ **d)** q, r **e)** c **f)** p **g)** f, b, a **h)** e, f, l, m, n **5.** No **7.** Level 0: a; level 1: b, c, d; level 2: e through k (in alphabetical order); level 3: l through r; level 4: s, t; level 5: u **9. a)** The entire tree **b)** c, g, h, o, p and the four edges cg, ch, ho, hp **c)** e alone **11. a)** 1 **b)** 2 **13. a)** 3 **b)** 9 **15.** The "only if" part is Theorem 2 and the definition of a tree. Suppose G is a connected simple graph with n vertices and $n - 1$ edges. If G is not a tree, it contains, by Exercise 14, an edge whose removal produces a graph G', which is still connected. If G' is not a tree, remove an edge to produce a connected graph G''. Repeat this procedure until the result is a tree. This requires at most $n - 1$ steps because there are only $n - 1$ edges. By Theorem 2, the resulting graph has $n - 1$ edges because it has n vertices. It follows that no edges were deleted, so G was already a tree. **17.** 9999 **19.** 2000 **21.** 999 **23.** 1,000,000 dollars **25.** No such tree exists by Theorem 4 because it is impossible for $m = 2$ or $m = 84$.

27. Complete binary tree of height 4:



Complete 3-ary tree of height 3:



29. a) By Theorem 3 it follows that $n = mi + 1$. Because $i + l = n$, we have $l = n - i$, so $l = (mi + 1) - i = (m - 1)i + 1$. **b)** We have $n = mi + 1$ and $i + l = n$. Hence, $i = n - l$. It follows that $n = m(n - l) + 1$. Solving for n gives $n = (ml - 1)/(m - 1)$. From $i = n - l$ we obtain $i = [(ml - 1)/(m - 1)] - l = (l - 1)/(m - 1)$. **31.** $n - t$ **33. a)** 1 **b)** 3 **c)** 5 **35. a)** The parent directory **b)** A subdirectory or contained file **c)** A subdirectory or contained file in the same parent directory **d)** All directories in the path name **e)** All subdirectories and files continued in the directory or a subdirectory of this directory, and so on **f)** The length of the path to this directory or file **g)** The depth of the system, i.e., the length of the longest path **37.** Let $n = 2^k$, where k is a positive integer. If $k = 1$, there is nothing to prove because we can add two numbers with $n - 1 = 1$ processor in $\log 2 = 1$ step. Assume we can add $n = 2^k$ numbers in $\log n$ steps using a tree-connected network of $n - 1$ processors. Let $x_1, x_2, \ldots, x_{2n}$ be $2n = 2^{k+1}$ numbers that we wish to add. The tree-connected network of $2n - 1$ processors consists of the tree-connected network of $n - 1$ processors together with two new processors as children of each leaf. In one step we can use the leaves of the larger network to find $x_1 + x_2, x_3 + x_4, \ldots, x_{2n-1} + x_{2n}$, giving us n numbers, which, by the inductive hypothesis, we can add in $\log n$ steps using the rest of the network. Because we have used $\log n + 1$ steps and $\log(2n) = \log 2 + \log n = 1 + \log n$, this completes the proof. **39.** c only **41.** c and h **43.** Suppose a tree T has at least two centers. Let u and v be distinct centers, both with eccentricity e, with u and v not adjacent. Because T is connected, there is a simple path P from u to v. Let c be any other vertex on this path. Because the eccentricity of c is at least e, there is a vertex w such that the unique simple path from c to w has length at least e. Clearly, this path cannot contain both u and v or else there would be a simple circuit. In fact, this path from c to w leaves P and does not return to P once it, possibly, follows part of P toward